The Planck distribution

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Abstract

The Planck radiation law can be viewed as a probability distribution of photon energies. This resource note is about its properties as a probability distribution. The distribution is intimately linked to many of the most famous special functions.

1 Background

For blackbody radiation the rate of emission of energy per unit area per unit time (i.e. the exitance) per unit frequency interval is given by Planck's radiation law:

$$B(\nu, T) = \frac{2\pi h}{c^2} \frac{\nu^3}{e^{h\nu/k_B T} - 1} \tag{1}$$

This can be viewed as a probability distribution of photon frequencies (or, equivalently after a rescaling, energies, or, after inversion, wavelengths).

Physically, the Planck distribution is singled out as being the maximum entropy distribution of energies of a 3D gas of photons at thermal equilibrium. In D space dimensions, the exponent in the numerator of equation 1 becomes D rather than 3. [CC05]. These (and further) generalisations of the distribution are discussed in [NK06, Pak21], where the distribution at hand becomes a member of a larger family of distributions.

It seems likely that this probability distribution is not more well known as a probability distribution (rather than as a physical quantity) due to the need of representing its properties using special functions. More recently computer algebra systems have made manipulating it more easy [Ste12].

2 Density function

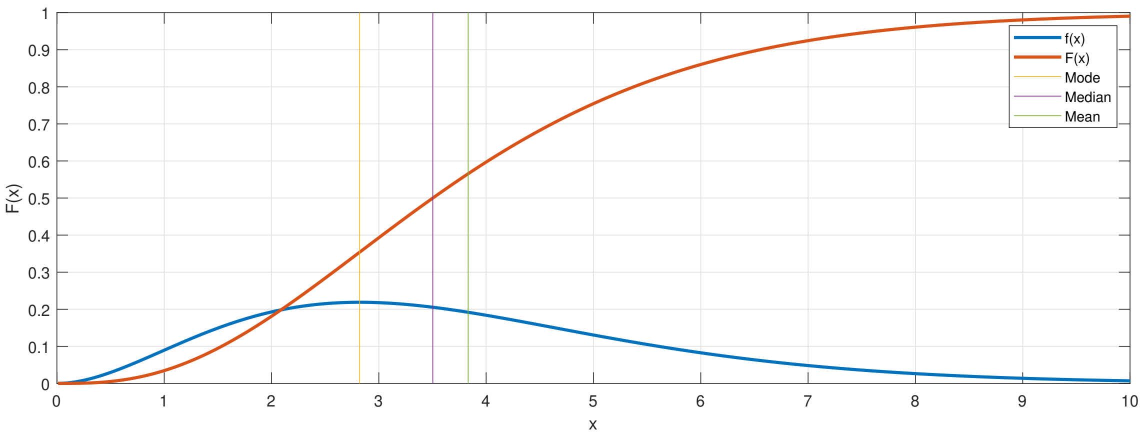

Figure 1: The PDF, CDF, mode, median and mean of the Planck distribution for $a = 1$.

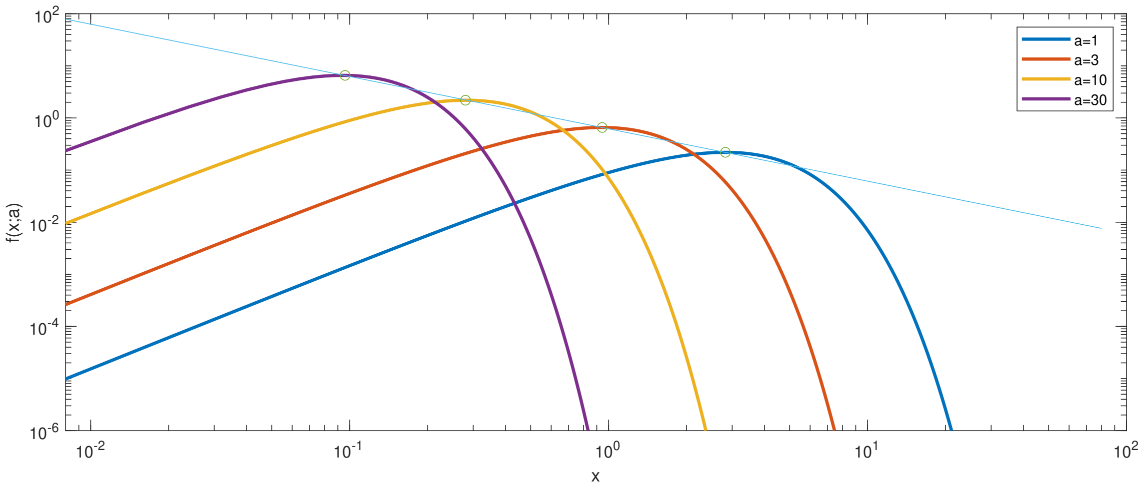

Figure 2: The PDFs of the Planck distribution for $a = 1, 3, 10$ and 30 in a log-log diagram. The maxima are connected by a line representing the rescaling relation 3.

Normalizing equation 1 into a probability distribution using[1]

$$\int_0^\infty \frac{\nu^3}{e^{a\nu}-1} d\nu = \frac{\pi^4}{15a^4}$$

we get a PDF

$$f(x;a) = \frac{15a^4}{\pi^4} \frac{x^3}{e^{ax}-1} \tag{2}$$

where $a = h/k_B T$ is the temperature-dependent shape and location parameter. $a \to 0$ corresponds to infinite temperature, $a \to \infty$ zero temperature.

2.1 Rescaling

The different temperature distributions are identical under a rescaling of x and the distribution.

$$f(x;a) = af(ax;1) \tag{3}$$

2.2 Tails

As can be seen in figure 2, $f(x;a) \sim x^2/a$ as $x \to 0$ (the Rayleigh–Jeans law). As $x \to \infty$ it declines as $f(x;a) \sim e^{-ax}$ (Wien's approximation).

3 Mode

The mode is given by $f'(x;a) = 0$ or

$$e^{ax}(ax-3)+3=0$$

for $x > 0$. This has the solution

$$x_{mode} = \frac{W(-3/e^3)+3}{a} \approx \frac{2.82144}{a} \tag{4}$$

where W is the Lambert W function. This is Wien's displacement law in physics.

[1]This is due to the beautiful identity

$$\int_0^\infty \frac{x^{s-1}}{e^{ax}-1} dx = \frac{\zeta(s)\Gamma(s)}{a^s}$$

and that the values of $\zeta(x)$ for even positive integers $2n$ are

$$\zeta(2n) = (-1)^{1+n} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$$

where B_n are the Bernoulli numbers.

4 Cumulative distribution function

The CDF can be found by integration by parts into a series of polylog functions [Ste12].

$$F(x) = \frac{15a^4}{\pi^4}\int_0^x \frac{t^3}{e^{at}-1}dx =$$
$$\frac{15a^4}{\pi^4}\left[\frac{360\,\mathrm{Li}_4\left(\mathrm{e}^{ax}\right) - 360ax\,\mathrm{Li}_3\left(\mathrm{e}^{ax}\right) + 180a^2x^2\,\mathrm{Li}_2\left(\mathrm{e}^{ax}\right) + 60a^3x^3\ln\left(1-\mathrm{e}^{ax}\right) - 15a^4x^4 - 4\pi^4}{60a^4}\right] =$$
$$\frac{1}{\pi^4}\left[90\,\mathrm{Li}_4\left(\mathrm{e}^{ax}\right) - 90\,\mathrm{Li}_3\left(\mathrm{e}^{ax}\right) + 45a^2x^2\,\mathrm{Li}_2\left(\mathrm{e}^{ax}\right) + 15a^3x^3\ln\left(1-\mathrm{e}^{ax}\right) - (15/4)a^4x^4 - \pi^4\right] \quad (5)$$

Note that the different terms generate complex values that cancel; however, for practical numeric work integral quadrature is likely more numerically stable.

5 Median

The median, $F(x_m) = 1/2$, does not appear to have any closed form expression. Numerically is is $\approx 3.503/a$.

6 Mean

The mean of the distribution is

$$\mu = E[X] = \int_0^\infty xf(x;a)dx = \frac{\zeta(5)\Gamma(5)}{\zeta(4)\Gamma(4)a} = \frac{360\zeta(5)}{\pi^4 a} \approx \frac{3.8322}{a}. \quad (6)$$

The ratio between mean and mode is hence constant: $\mu/x_{mode} = \pi^4(W(-3/e^3)+3)/360\zeta(5) \approx 1.3583$ (the same constant ratio property is true for the median, located between them).

7 Variance

The variance becomes

$$\sigma^2 = E[X^2] - E[X]^2 = \frac{\zeta(6)\Gamma(6)}{\zeta(4)\Gamma(4)a^2} - \left(\frac{\zeta(5)\Gamma(5)}{\zeta(4)\Gamma(4)a}\right)^2 =$$
$$\frac{15a^4}{\pi^4}\frac{8\pi^6}{63a^6} - \frac{360^2\zeta(5)^2}{\pi^8 a^2} = \left(\frac{40\pi^2}{21} - \frac{360^2\zeta(5)^2}{\pi^8}\right)\frac{1}{a^2} \approx \frac{4.1133}{a^2} \quad (7)$$

8 Skewness and higher order moments

The skewness can be expressed as

$$\mu_3 = \frac{E[X^3] - 3\mu\sigma^2 - \mu^3}{\sigma^3} \quad (8)$$

where

$$E[X^3] = \frac{\zeta(7)\Gamma(7)}{\zeta(4)\Gamma(4)a^3} \approx \frac{111.7983}{a^3}. \quad (9)$$

This gives $\mu_3 \approx 8.2293/a^3$.

Higher order (raw) moments can be calculated similarly,

$$E[X^n] = \int_0^\infty \frac{x^{3+n}}{e^{ax}-1}dx = \frac{\zeta(4+n)\Gamma(4+n)}{\zeta(4)\Gamma(4)a^{n+1}}. \quad (10)$$

8.1 Moment-generating function

The moment generating function is

$$M(t) = E[e^{tX}] = \frac{15a^4}{\pi^4}\int_0^\infty x^3 \frac{e^{tx}}{e^{ax}-1}dx. \tag{11}$$

This can be converted into a polygamma function using the definition[2]

$$\psi^{(3)}(z) = \int_0^\infty \frac{x^3 e^{-zt}}{1-e^{-x}}dx. \tag{12}$$

Hence,

$$M(t) = \frac{15}{\pi^4}\psi^{(3)}(1+t/a). \tag{13}$$

9 Sums

The sum of two Planck distributed variates is not Planck distributed. The low-frequency tail has a higher exponent than the vanilla Planck distribution.

Performing a convolution to find $f_{X+Y}(z) = \int_0^z f(x)f(z-x)dx$ produces the non-illuminating

$$\begin{aligned} f_{X+Y}(z) = \frac{1350a}{\pi^8(e^{az}-1)}&[120\,\mathrm{Li}_7\left(\mathrm{e}^{az}\right) - 60az\,\mathrm{Li}_6\left(\mathrm{e}^{az}\right) + 12a^2z^2\,\mathrm{Li}_5\left(\mathrm{e}^{az}\right) - a^3z^3\,\mathrm{Li}_4\left(\mathrm{e}^{az}\right) \\ &+ 120\,\mathrm{Li}_7\left(\mathrm{e}^{-az}\right) + 60az\,\mathrm{Li}_6\left(\mathrm{e}^{-az}\right) + 12a^2z^2\,\mathrm{Li}_5\left(\mathrm{e}^{-az}\right) + a^3z^3\,\mathrm{Li}_4\left(\mathrm{e}^{-az}\right) \\ &- 24\,\zeta\,(5)\;a^2z^2 - 240\,\zeta\,(7)] \end{aligned} \tag{14}$$

As $k \to \infty$ independent variates are added, the central limit theorem makes their sum converge (since they have finite variance) to $\to N(k\mu, k\sigma^2)$. Since the third moment $E[X^3] = \rho$ is finite, by the Berry–Esseen theorem the difference between the distribution of the sample sums and the relevant Gaussian is bounded by

$$\sup|F_k(x) - \Phi(x)| \le \frac{C\rho}{\sigma^3\sqrt{k}} \tag{15}$$

(where $C < 0.4748$ [She11]). This gives a bound here scaling as $0.4684/\sqrt{k}$, independent of a.

10 Entropy

The physical entropy of the distribution is described and analysed by Alfonso Delgado-Bonal in [DB17]. See also [ASV16] for a discussion of the physical meaning of the entropy carried by each photon.

The differential entropy $H = -\int_0^\infty f(x)\log(f(x))dx$ does not have a closed form expression. It is a growing function of a.

11 Sampling

Sampling from the distribution $f(x,1)$ has been described by Charles Barnett and Eugene Canfield in [BC70], using two methods.

First, they described a series expansion method using

$$f(x) = \frac{15}{\pi^4}x^2e^{-x}\sum_{k=0}^{\infty}e^{-kx} = \sum_{n=1}^{\infty}\left(\frac{90}{\pi^4n^4}\right)\left(\frac{n^4}{6}x^3e^{-nx}\right) = \sum_{n=1}^{\infty}\pi_n f_n(x) \tag{16}$$

where $\pi_n = 90/\pi^4n^4$ can be viewed as a probability of selecting a sample from the gamma distribution function $f_n(x) = n^4x^3e^{-nx}/6$; the full distribution is a weighted mix of these distributions.

[2]Another approach is to convert the fraction into a geometric series expansion, reorder the sum and integral, and arriving at a Hurwitz zeta function that can be converted to a polygamma since it has an integer exponent.

Second, they described a rejection technique where the distribution is bounded by $h(x) = 6.25(15/\pi^4)x/(1-e^{-x})(x+e^{-x})$. In both cases it is necessary to sample from random distributions of the form $f_n(x)$.

Alice Graf Brolund and Rebecca Persson used a far simpler (but inexact) rejection sampling using a uniform distribution bounded by a multiple of the peak frequency [GBP18].

Generally the series expansion method is fast and effective.

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